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Translated by Z.L.

J. Appl. Maths Mechs, Vol. 55, No. 2, pp. 176-180, 1991
Printed in Great Britain

0021-8928/91 $15.00+0.00

ACCESSORY PARAMETERS IN CIRCULAR QUADRANGLES*

P. YA. POLUBARINOVA-KOCHINA

The problem of the conformal mapping of a circular polygon in a half-plane (a half-strip, a rectangle, a circle, etc.) is of considerable importance, e.g. in the theory of groundwater motion. The question of accessory (redundant) parameters that may arise in this case is not trivial and deserves special analysis. Some special cases of such problems are considered in this paper.

1. A circular triangle in a plane is completely defined by the position of its three vertices and the three angles at the vertices. For a circular quadrangle, the specification of three of its vertices and the four angles does not completely define the quadrangle, and the fourth vertex may have an infinite set of positions /1, p.306/. Let us consider this problem in more detail for the case of a circular rectangle (Fig.1) with given sides $A_2A_3 = a$ and $A_1A_2 = b$. Draw two families of auxiliary circles tangent to the segments A_1A_4 and A_3A_4 at the points A_1 and A_3, respectively. An infinite set of these circles intersects at a right angle (points A and A'). We will show that the family of such points $A\ (x, y)$ is a circle through the vertices of the rectangle $A_1A_2A_3A_4$ (Fig.1).

Let $(0, b_1)$ and $(a_1, 0)$ be the centres of two circles through the point A. Then the equations of these circles are

$$x^2 + (y - b_1)^2 = (b_1 - b)^2,\ (x - a_1)^2 + y^2 = (a_1 - a)^2 \tag{1.1}$$

The expressions for the slope of the tangents at the point A

$$y_1' = -x/(y - b_1),\quad y_2' = -(x - a_1)/y$$

* *Prikl.Matem.Mekhan.*,55,2,222-227,1991

and the orthogonality condition $y_1' y_2' = -1$ give

$$x(x - a_1) + y(y - b_1) = 0 \tag{1.2}$$

Eliminating a_1 and b_1 from relationships (1.1)-(1.2), we obtain the equation

$$ay^3 + b(x - 2a)y^2 + (x - a)(ax - b^2)y + bx(x - a)^2 = 0 \tag{1.3}$$

Substituting $x = r\cos\theta$, $y = r\sin\theta$, we can obtain a quadratic equation for r. This is not needed, however, and we will return to Eq.(1.3) at the end of the next section.

2. A conformal mapping of a circular quadrangle with the angles $\pi\alpha, \pi\beta, \pi\gamma, \pi\delta$ (Fig.2) in a half-plane may be analysed using a Fuchs differential equation

$$Y'' + \left(\frac{1-\alpha}{\zeta} + \frac{1-\beta}{\zeta - 1} + \frac{1-\gamma}{\zeta - e}\right)Y' + \frac{(\varepsilon\varepsilon'\zeta + \lambda)Y}{\zeta(\zeta-1)(\zeta-e)} = 0 \tag{2.1}$$

The Fuchs relationships must hold in this case:

$$\alpha + \beta + \gamma + \varepsilon + \varepsilon' = 2, \quad \varepsilon - \varepsilon' = \delta \tag{2.2}$$

The vertices of the quadrangle correspond to the following points in the plane ζ: $\zeta = 0, 1, e, \infty$ (Fig.2).

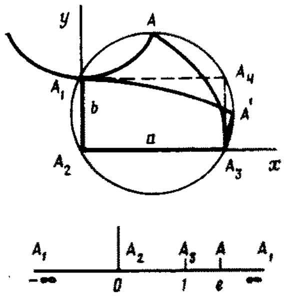

Fig.1

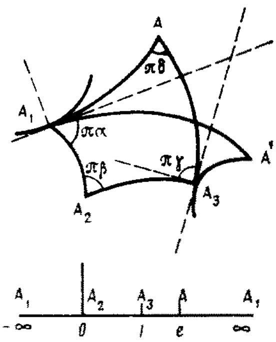

Fig.2

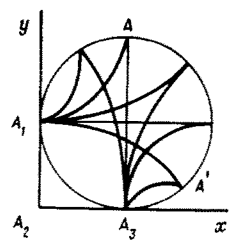

Fig.3

The real number λ is an accessory coefficient or parameter. For a rectangle,

$$\varepsilon\varepsilon' = 0, \quad \lambda = 0$$

If the linearly independent integrals y_1, y_2 of Eq.(2.1) have been found, then the conformal mapping of the quadrangle $A_1A_2A_3A$ from the z plane to the ζ plane is defined by the equation

$$z = x + iy = (C_1y_1 + C_2y_2)/(C_3y_1 + C_4y_2) \tag{2.3}$$

The ratios of the three arbitrary constants C_i to the fourth constant (in general, these are complex numbers) are sufficient to determine three vertices of the quadrangle, A_1, A_2 and A_3, say. For the fourth vertex, A say, we have two real numbers λ and e. One of these parameters is redundant (the fourth vertex must lie on a certain curve, so that a single parameter is sufficient). Therefore, λ and e are dependent.

Let us consider the example of a circular quadrangle with right angles. In this case, $\alpha = \beta = \gamma = \delta = 1/2$, $\varepsilon = \delta$, $\varepsilon' = 0$, and Eq.(2.1) is rewritten in the form

$$Y'' + \frac{1}{2}\left(\frac{1}{\zeta} + \frac{1}{\zeta - 1} + \frac{1}{(\zeta - e)}\right)Y' + \frac{\lambda Y}{\zeta(\zeta-1)(\zeta-e)} = 0 \tag{2.4}$$

A fractional-linear transformation takes two neighbouring circles to two straight lines, so that we are dealing with a right quadrangle $A_1A_2A_3A$ (Fig.1).

As the fundamental system of solutions of Eq.(2.4), we take the expressions /2, 3/

$$y_1 = \cos[2k\sqrt{\lambda}F_*(\zeta, k)], \quad y_2 = \sin[2k\sqrt{\lambda}F_*(\zeta, k)] \quad (k = \sqrt{e})$$

$$F_*(\zeta, k) = F(\arcsin\sqrt{\zeta}, k) = \int_0^{\sqrt{\zeta}} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}$$

The condition $z = 0$ for $\zeta = 0$ gives $C_1 = 0$ in (2.3). Assume $C_2 = 1$ and introduce the notation

$$\mu = 2k\sqrt{\lambda}K,\ \nu = 2k\sqrt{\lambda}K'$$

$$K = \int_0^1 \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}},\quad K' = \int_0^1 \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k'^2\zeta^2)}}\quad (k' = \sqrt{1-k^2})$$

Considering the points A_2 and A_3, we have

$$a = \frac{\sin\mu}{C_3\cos\mu + C_4\sin\mu},\quad b = \frac{\operatorname{sh}\nu}{C_3\operatorname{ch}\nu + iC_4\operatorname{sh}\nu}$$

For a and b to be real, we must take $C_4 = 0$. Then for C_3 we obtain two expressions

$$C_3 = \operatorname{tg}\mu/a = \operatorname{th}\nu/b \tag{2.5}$$

Hence we find the relationship between λ and $e = 1/k^2$ in the form

$$b/a = \operatorname{th}(2k\sqrt{\lambda}K')/\operatorname{tg}(2k\sqrt{\lambda}K) \tag{2.6}$$

In the fourth vertex $A(x, y)$ we have

$$x + iy = \frac{a}{\operatorname{tg}\mu}\operatorname{tg}(\mu + i\nu) = \frac{a}{\operatorname{tg}\mu}\frac{\operatorname{tg}\mu + i\operatorname{th}\nu}{1 - i\operatorname{tg}\mu\operatorname{th}\nu} \tag{2.7}$$

Using (2.5) to eliminate $\operatorname{th}\nu$ in Eqs.(2.7), we obtain

$$x = \frac{a(a^2 - b^2\operatorname{tg}^2\mu)}{a^2 + b^2\operatorname{tg}^2\mu},\quad y = \frac{ab(1 + \operatorname{tg}^2\mu)}{a^2 + b^2\operatorname{tg}^2\mu} \tag{2.8}$$

If we eliminate $\operatorname{tg}\mu$ from Eqs.(2.8), we obtain the equation of a circle

$$x^2 + y^2 - ax - by = 0 \tag{2.9}$$

through the points A_1, A_2, A_3 and A_4 (Fig.1).

3. We can now return to the cubic Eq.(1.3) and conjecture that its left-hand side is divisible by the left-hand side of (2.9). Indeed, (1.3) decomposes into the equation of a circle (2.9) and the equation of a line

$$bx + ay - ab = 0 \tag{3.1}$$

i.e., the equation of the diagonal A_1A_3 and its continuations. The line (3.1) is the locus of the second point of intersection of the auxiliary circles (1.1).

Note that the condition of orthogonality of the auxiliary circles (1.1) at the point A (Fig.1) may be replaced with the condition that these circles intersect at an arbitrary angle θ. Then we should have the equation

$$\operatorname{tg}\theta = (y_1' - y_2')/(1 + y_1'y_2')$$

In particular, for $\theta = 0$ or $y_1' = y_2'$, we obtain an equation of the locus of the points A:

$$2x(x-a)(r^2-b^2) + 2y(y-b)(r^2-a^2) = (r^2-a^2)(r^2-b^2),\quad r^2 = x^2 + y^2 \tag{3.2}$$

This equation splits into the equations of two circles, which may be represented in the form

$$\left(x - \frac{a+b}{2}\right)^2 + \left(y - \frac{a+b}{2}\right)^2 = \frac{a^2+b^2}{2} \tag{3.3}$$

$$\left(x - \frac{a-b}{2}\right)^2 + \left(y - \frac{b-a}{2}\right)^2 = \frac{a^2+b^2}{2} \tag{3.4}$$

The circle (3.3) is the locus of zero vertices; an arc of the circle (3.4) is the boundary of the circular arcs forming the required quadrangle.

For $a = b$, we have the circles (Fig.3)

$$(x-a)^2 + (y-a)^2 = a^2,\ x^2 + y^2 = a^2$$

For an arbitrary quadrangle without angles equal to π or 2π, we can find the locus of the vertices A for given A_1, A_2, A_3, as shown in Fig.2. However, the equation relating λ and e (or the elimination of λ) must be considered specially and by a different technique in each case.

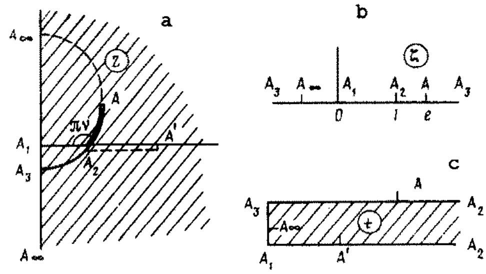

Fig.4

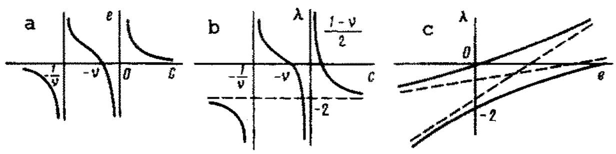

Fig.5

4. In many problems of the motion of groundwater with a free surface (in earth dams) or with an interface separating fresh and saline water, etc., the velocity hodograph is a circular polygon with (circular or straight) cuts. As an example, Fig.4 shows a circular cut ending at the point A or a straight cut ending at the point A_1. If the affix of the point A (or A' is denoted by e, then for $0 < e < 1$ we have a straight cut, for $e = 1$ there is no cut, and for $e > 1$ the cut starts propagating along the circular arc A_2A. Bereslavskii /4/ obtained closed-form exact solutions for polygons with cuts, i.e., with an angle 2π at one of the vertices and a number of right angles*. For the case shown in Fig.4, Eq.(2.1) reduces to the form

$$Y'' + \left(\frac{1}{2\zeta} + \frac{1-\nu}{\zeta-1} - \frac{1}{\zeta-e}\right)Y' + \frac{\nu(\nu+1)\zeta+\lambda}{4\zeta(\zeta-1)(\zeta-e)}Y = 0 \tag{4.1}$$

Linearly independent partial solutions of this equation were obtained in the form

$$y_1 = \frac{\operatorname{ch} t \operatorname{ch} \nu t + C \operatorname{sh} t \operatorname{sh} \nu t}{\operatorname{ch}^{1+\nu} t} = \frac{\operatorname{ch} \nu t}{\operatorname{ch}^{\nu} t}(1 + C \operatorname{th} t \operatorname{th} \nu t) \tag{4.2}$$

$$y_2 = \frac{\operatorname{ch} t \operatorname{sh} \nu t + C \operatorname{sh} t \operatorname{ch} \nu t}{\operatorname{ch}^{1+\nu} t} = \frac{\operatorname{ch} \nu t}{\operatorname{ch}^{\nu} t}(\operatorname{th} \nu t + C \operatorname{th} t)$$

Here C is a parameter to be discussed later, t is a new variable related to ζ by the equation $\zeta = \operatorname{th}^2 t$.

The domain of the variable $t = t' + it''$ is a half-strip (Fig.4c).

For the conformal mapping of this polygon, we can substitute the integrals (4.2) into expression (2.3) and to determine the constants C_i. Examining the conditions at the vertices A_1, A_2, A_3, we obtain

$$z = x + iy = a\,(\operatorname{th} \nu t + C \operatorname{th} t)/(1 + C \operatorname{th} \nu t \operatorname{th} t) \tag{4.3}$$

The parameter C is related to the parameter e of Eq.(4.1) by the equality

$$e = (C + \nu)/[C\,(1 + C\nu)] \tag{4.4}$$

The accessory parameter λ depends on C:

$$\lambda = \nu\,(\nu + C)\,(1 - \nu - 2C)/[C\,(1 + C\nu)] \tag{4.5}$$

If we eliminate C between (4.4) and (4.5), we obtain a quadratic equation for λ and e (see Fig.5):

$$\lambda^2 - 2\,[(\nu^2 - \nu - 1)\,e + 1]\,\lambda + e^2\nu\,(\nu^2 - 1)\,(\nu - 2) - 2\nu\,(\nu + 1)\,e = 0 \tag{4.6}$$

*See also BERESLAVSKII E.N., Mathematical Modelling of Seepage Flow with Free Boundaries Doctorate Dissertation, Kazan, 1990.

An equation of the same form (for any $\alpha, \beta, \gamma, \delta$ with one of them equal to 2) was obtained in /5, 6/ by different techniques, Here we take, for simplicity, the neighbourhood of the point $\zeta = 0$ and seek the solution of Eq.(2.1) in series form $y_1 = a_0 + a_1\zeta + a_2\zeta^2 + \ldots$. Substituting into Eq.(2.1), we obtain

$$a_0\lambda + a_1(1-\alpha)e = 0$$
$$\varepsilon\varepsilon' a_0 + a_1[\lambda - (1-\alpha)(1+e) - (1-\beta)e - (1-\gamma)] + 2a_2 e(2-\alpha) = 0$$

Hence, for $a_0 \neq 0$, setting $\alpha = 2$, we obtain an equation for λ:

$$\lambda^2 + (1+e)\lambda + (1-\beta)e + 1 - \gamma - \varepsilon\varepsilon' e = 0$$

The coefficient a_2 remains undetermined and it may be used as the second arbitrary constant in the solution y_1.

Returning to Bereslavskii's problem, we note that the parameter C turned out to be a uniformizing variable (see /1, p.414/) for Eq.(4.6), producing single-valued representations of the variables e and λ.

Let us check the correctness of solution (4.3). Along the line A_1AA_2 we have $t = t'$; the right-hand side of equality (4.3) is real and therefore $y = 0$, which is correct.

To pass to the line A_2A_3 in the plane t, set $t = t' + \frac{\pi}{2}i$. Then we have

$$\operatorname{th} t = \operatorname{cth} t', \quad \operatorname{th} \nu t = \frac{\operatorname{th} \nu t' + i\mu}{1 + i\mu \operatorname{th} \nu t'} \left(\mu = \operatorname{tg} \frac{\pi\nu}{2} = \frac{a}{b}\right)$$

For z we obtain instead of (4.3)

$$z = a\frac{P + i\mu}{1 + i\mu P}, \quad P = \frac{\operatorname{th} \nu t' + C \operatorname{cth} t'}{1 + C \operatorname{th} \nu t' \operatorname{cth} t'} \tag{4.7}$$

Using (4.7), we have

$$y = \mu a \frac{1 - P^2}{1 + \mu^2 P^2}, \quad x^2 + y^2 = a^2 \frac{P^2 + \mu^2}{1 + \mu^2 P^2}$$

whence we obtain the equation of the circle containing the arc A_3A_2A:

$$x^2 + y^2 - [(a^2 - b^2)/b]\, y = a^2$$

It remains to consider the line $A_1A_\infty A_3$. On this line, $t = it'$ and the right-hand side of Eq.(4.3) is imaginary; at the point A_∞ (Fig.4) we have

$$\operatorname{tg} \nu t'' \operatorname{tg} t'' = -1/C, \; y = \infty$$

At the point A_3, where

$$\operatorname{tg} \nu t = i \operatorname{tg} {}^1/_2 \nu\pi, \; \operatorname{th} t = i \operatorname{tg} {}^1/_2 \pi = \infty$$

we obtain $z = aC/(iC \operatorname{tg} {}^1/_2 \nu\pi) = -ib$.

We have thus shown that (4.3) indeed solves the problem of the conformal mapping of a quadrangle (Fig.4).

This series example from Bereslavskii's problem is remarkable not only in the simplicity of the solution but also in the fact that the parameter λ does not occur in the solution.

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